EQUILIBRIUM CRACKS FORMED DURING BRITTLE FRACTURE RECTILINEAR CRACKS IN PLANE PLATES

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The basic concepts of the theory of equilibrium cracks, i.e. cracks the dimensions of which do not change at a given load, have been discussed in [1]. In order to avoid numerous references and repetitions in the present paper, we reproduce here briefly the fundamental initial assumptions of [1]. The crack is subdivided into two regions: the *inner* region, in which the distance between the opposite edges of the crack is considerable and their interaction is negligibly small and the *end* region, in which the opposite edges of the crack are closely adjacent to each other and where cohesive forces of very considerable amount are acting. The entire concept is based upon the following three hypotheses:

1. *The longitudinal dimensions of the end region are small as compared with those of the entire crack.*

2. *The distribution of the displacements of the surface points of the end region of the crack does not depend on the acting loads and is always the same for the given material under given conditions.* The cohesive forces, which attract the opposite edges of the crack toward each other, depend only on the distribution of the displacements in the end region; therefore, the hypothesis just stated involves independence of these forces from the loads.

3. *The opposite edges of the crack are smoothly connected with each other at its ends, or, what is the same, the stress is finite at the ends of the crack.* This hypothesis was originally advanced by Khristianovich [2] in a study of the problems of formation and development of cracks in rock strata. The only compressing factor in these problems is the rock pressure, the pressure produced by the weight of the upper rock layers; the cohesive forces were not discussed and not taken into account.

The model suggested above was applied in [1] to the solution of the problem of axisymmetrical equilibrium cracks. In the present paper, the sameideas are being applied to the solution of the problem of rectilinear equilibrium cracks in plane plates. We will find that in the case now to be considered the dimensions of the crack are again determined by the applied loads and by the new universal characteristic coefficient of the material, the cohesion modulus K, introduced in [1]. The cohesive forces are of essential influence only on the dimensions of the cracks and on the distribution of the displacements of the opposite edges of the crack in the vicinity of its ends.

1. Condition of smooth connection and of finite stresses at the ends of a isolated slit [cut] in the infinite plane.

We have to start with the following problem. Consider a slit from point $x = a$ to point $x = b$ along the x-axis in the infinite plane (Fig. 1). Normal stresses $- g(x)$ and opposite direction are applied to points of opposite edges having the same coordinate x. Let us determine the conditions which must be fulfilled, if the stresses at the ends of the slit are to be finite, or, what is the same, if the opposite edges of the deformed surface of the slit are to be smoothly connected with each other at the ends of the slit.

We replace the coordinates x, y by the coordinates

$$x' = x - {}^1/_2(a+b), \qquad y' = y$$

so that the slit is symmetrical with respect to the origin of the new system of coordinates. For the solution of the problem under consideration we use the method of Muskhelishvili [4]. We recall the fundamental relations of this method:

$$X_x + Y_y = 4 \operatorname{Re}\{\Phi(\zeta)\} \qquad (1.1)$$

$$Y_y - X_x + 2iX_y = 2\left\{\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\Phi'(\zeta) + \Psi(\zeta)\right\} \qquad (1.2)$$

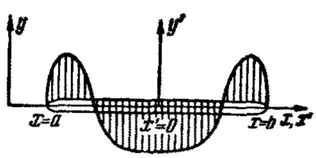

Fig. 1.

where X_x, Y_y, X_y are the components of the stress tensor, while the function

$$z' = x' + iy' = \omega(\zeta) = {}^1/_2 l(\zeta + \zeta^{-1}), \qquad l = {}^1/_2(b-a) \qquad (1.3)$$

yields the mapping of the exterior of the slit on the exterior of the unit circle on the parametric ζ-plane. The functions $\Phi(\zeta)$ and $\Psi(\zeta)$ are determined by the relations

$$\Phi(\zeta) = \frac{\varphi'(\zeta)}{\omega'(\zeta)}, \qquad \Psi(\zeta) = \frac{\psi'(\zeta)}{\omega'(\zeta)} \qquad (1.4)$$

with

$$\varphi(\zeta) = -\frac{1}{2\pi i}\int\frac{f d\sigma}{\sigma-\zeta}, \qquad \psi(\zeta) = -\frac{1}{2\pi i}\int\frac{\bar{f} d\sigma}{\sigma-\zeta} - \zeta\frac{1+\zeta^2}{\zeta^2-1}\varphi'(\zeta) \tag{1.5}$$

The integrals are to be taken along the periphery of the unit circle, while

$$f = \bar{f} = -\int_{x_0'}^{x'(\sigma)} g(x')\, dx' \tag{1.6}$$

where x_0' is a certain fixed point and $x'(\sigma)$ is a variable point. Since the function f is real, the formulas (1.3) to (1.5) show that the condition

$$\frac{\bar{\omega}}{\omega'}\Phi' + \Psi \equiv 0$$

is fulfilled along the entire x-axis, i.e. along the entire slit and its continuation. Thus, by virtue of (1.2) we have along the slit itself and along its continuation

$$X_x = Y_y, \qquad X_y = 0 \tag{1.7}$$

Consequently

$$X_x = Y_y = 2 \text{ Re}\{\Phi(\zeta)\} \tag{1.8}$$

Thus we find that the necessary and sufficient condition for obtaining finite stresses at the ends of the slit is fulfilled, if $\Phi(+1)$ and $\Phi(-1)$ are finite, since the points $\zeta = +1$ and $\zeta = -1$ correspond to the ends of the slit.

The simplest way of determining $\phi(\zeta)$ is as follows. If concentrated splitting forces P of equal amount and opposite direction are applied to the opposite edges of the slit at the point $x' = x_0'$ of the slit surface, then by virtue of (1.5) we can obtain

$$\varphi(\zeta) = \frac{P}{2\pi i}\ln\frac{\zeta - e^{i\lambda}}{\zeta - e^{-i\lambda}}, \qquad \lambda = \arccos\frac{x_0'}{l} \tag{1.9}$$

Considering that, with $x' = l\cos\lambda$ along the slit itself, the force $g(x')dx' = g(l\cos\lambda)\, l\sin\lambda\, d\lambda$ is acting on the element dx' of the slit, we find by summation of solutions (1.9) the following formula for the function $\phi(\zeta)$ in the case of the general problem under study

$$\varphi(\zeta) = \frac{l}{2\pi i}\int_0^\pi g(l\cos\lambda)\sin\lambda \ln\left(\frac{\zeta - e^{i\lambda}}{\zeta - e^{-i\lambda}}\right) d\lambda \tag{1.10}$$

whence

$$\Phi(\zeta) = \frac{\varphi'(\zeta)}{\omega'(\zeta)} = \frac{\zeta^2 N(\zeta)}{(\zeta^2-1)\pi}, \qquad N(\zeta) = 2\int_0^\pi \frac{g\sin^2\lambda\, d\lambda}{\zeta^2+1-2\zeta\cos\lambda} \tag{1.11}$$

To make the stresses at the end $x' = l$ of the slit finite, or, what is the same, to make $\Phi(1)$ finite, it is necessary that $N(1)$ should vanish, i.e.

$$\int_0^\pi \frac{g(l\cos\lambda)\sin^2\lambda\, d\lambda}{1-\cos\lambda} = 0 \tag{1.12}$$

Passing to the variable $x' = l\cos\lambda$ and to the variable $x' = x + 1/2(b + a)$, we find

$$l\int_0^\pi \frac{g\sin^2\lambda\, d\lambda}{1-\cos\lambda} = \int_{-l}^{l} \frac{g\sqrt{l^2-x'^2}dx'}{l-x'} = \int_{-l}^{l} g\sqrt{\frac{l+x'}{l-x'}}dx' = \int_a^b g(x)\sqrt{\frac{x-a}{b-x}}dx$$

Therefore, condition (1.12) for obtaining finite stresses at the right-hand end $x = b$ of the slit assumes the form

$$\int_a^b g(x)\sqrt{\frac{x-a}{b-x}}dx = 0 \tag{1.13}$$

Condition (1.13) for finite stresses can be obtained also immediately with the aid of the known method developed by Sedov [5] in the theory of thin wings.

In exactly the same way we derive the condition for obtaining finite stresses at the left-hand end $x = a$ of the slit in the form

$$\int_a^b g(x)\sqrt{\frac{b-x}{x-a}}dx = 0 \tag{1.14}$$

It is not difficult to show that these conditions are not only necessary, but also sufficient for obtaining finite stresses at the ends of the slit. We are now going to show that condition (1.12) and, consequently, condition (1.13), ensure smooth connection of the opposite edges on the slit at the end $x = b$. Indeed, in accordance with the formula of Kolosov-Muskhelishvili [4]

$$2\mu(u+iv) = \varkappa\varphi(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi'(\zeta)} - \overline{\psi(\zeta)} \tag{1.15}$$

where u and v are the components of displacement in the x and y directions, respectively, while μ is the shear modulus, ν Poisson's ratio and $\kappa = 4 - 3\nu$. From (1.15), together with (1.3), (1.5), we obtain for ζ-values with an absolute magnitude equal to unity (such ζ-values correspond to the contour of the slit)

$$2\mu(u+iv) = \varkappa\varphi(\zeta) - \overline{\varphi(\zeta)}, \qquad v = \frac{\varkappa+1}{2\mu}\operatorname{Im}[\varphi(\zeta)]. \tag{1.16}$$

For x but slightly differing from b, i.e. for ζ determined by the relation $\zeta = e^{i\theta}$, where θ is a small quantity, we find

$$\varphi(\zeta) = \frac{l}{2\pi i}\int_0^\pi g(l\cos\lambda)\sin\lambda\ln\left(\frac{1-e^{i\lambda}}{1-e^{-i\lambda}}\right)d\lambda -$$

$$-\frac{l\theta}{2\pi i}\int_0^\pi \frac{g(l\cos\lambda)\sin^2\lambda\, d\lambda}{1-\cos\lambda} - \frac{2l\theta^2}{\pi}\int_0^\pi \frac{g(l\cos\lambda)\sin^2\lambda d\lambda}{1-\cos\lambda} + O(\theta^3)$$

In so far as the quantity $\ln[(1-e^{i\lambda})/(1-e^{-i\lambda})]$ is purely imaginary, we obtain

$$v = \frac{l\varkappa(\varkappa+1)}{4\pi\mu}\int_0^\pi \frac{g(l\cos\lambda)\sin^2\lambda d\lambda}{1-\cos\lambda} + O(\theta^3) \tag{1.17}$$

In the case of small θ we have $x' = l(1-1/2\,\theta^2)$ by virtue of (1.3), so that $dx'/d\theta = dx/d\theta = -l\theta$, and we find in the vicinity of the right-hand end of the slit

$$\frac{dv}{dx} = \frac{dv}{d\theta}\frac{d\theta}{dx} = -\frac{\varkappa+1}{4\pi\mu\theta}\int_0^\pi \frac{g(l\cos\lambda)\sin^2\lambda\, d\lambda}{1-\cos\lambda} + O(\theta) \tag{1.18}$$

Thus, a smooth connection of the opposite edges at the right-hand end, corresponding to $\theta = 0$, takes place only if condition (1.12), or, what is the same, if condition (1.13) is fulfilled, and this is what we intended to prove. In an exactly similar way we can prove that the condition of smooth connection of the opposite edges of the slit at the left-hand end $x = a$ of the slit is identical with condition (1.14).

2. General investigation of a rectilinear equilibrium crack in an infinite plane plate. Consider an infinite plane plate acted upon by a tensile loading, symmetrical with respect to some straight line, the axis of symmetry (Fig. 2). If we disregard the elements of accident, then the plate must split along the axis of symmetry. Assume that a finite loading is applied to the plate on each side of the axis of symmetry, then the result of the process indicated is a rectilinear slit, which reaches some definite dimensions, the coordinates of the crack ends being $x = a$ and $x = b$; the crack itself remains invariable, if the loading remains constant.

The state of stress in a plate with a crack can be conveniently represented as a sum of two states of stress, one of which corresponds to the infinite plate, without a crack, under the given tensile loading, while the other corresponds to a plate with a crack, over the surface of which shear stress resultants and cohesive forces are acting. In the resulting state of stress the inner part of the crack is free of stress, while cohesive forces are acting in the end region; therefore, the intensity of the compressive stress resultants, responsible for the destruction of the plate, of the second state of stress equals in magnitude and is opposite in direction to that of the tensile stresses of the first state along the axis of symmetry. For the first state the displacements of the points of the axis of symmetry are zero, therefore these displacements are fully determined by the second state of stress. The latter corresponds to the conditions stated in the preceding section, the distribution of the stresses $g(x)$ being determined by

$$g(x)=\begin{cases} p(x)-G(x) & (a\leqslant x\leqslant a+d)\\ p(x) & (a+d\leqslant x\leqslant b-d)\\ p(x)-G(x) & (b-d\leqslant x\leqslant b)\end{cases} \tag{2.1}$$

where $p(x)$ is the intensity of the normal tensile stresses at the axis of symmetry for the first state of stress, while $G(x)$ is the intensity of the cohesive forces and d is the width of the end region*.

In consequence of our hypothesis of smooth connection of opposite edges of the crack at its ends (third hypothesis of our system, or the hypothesis of Khristianovich) the conditions

$$\int_a^b g(x)\sqrt{\frac{x-a}{b-x}}\,dx=0,\qquad \int_a^b g(x)\sqrt{\frac{b-x}{x-a}}\,dx=0 \tag{2.2}$$

must be fulfilled.

Take, for example, the first of conditions (2.2), the condition which ensures appearance of finite stresses and smooth connection at the end $x = b$ of the crack and substitute into it expression (2.1) for the distribution of the stresses $g(x)$. We find

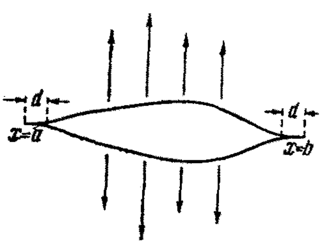

Fig. 2.

$$\int_a^b g(x)\sqrt{\frac{x-a}{b-x}}\,dx=\int_a^b p(x)\sqrt{\frac{x-a}{b-x}}\,dx-\\ -\int_{b-d}^{b} G(x)\sqrt{\frac{x-a}{b-x}}\,dx-\int_a^{a+d} G(x)\sqrt{\frac{x-a}{b-x}}\,dx \tag{2.3}$$

* The function $p(x)$ can be calculated in an elementary way for a given load; therefore it can be considered as given.

Consider the second and the third integrals. According to the first hypothesis $d \ll b$, so that we can assume

$$I_2 = \int_{b-d}^{b} G(x)\sqrt{\frac{x-a}{b-x}}\,dx \approx \sqrt{2l}\int_{b-d}^{b}\frac{G(x)\,dx}{\sqrt{b-x}}, \qquad l = \frac{b-a}{2}$$

Passing to the variable $s = b - x$, measured from the end $x = b$ of the crack and replacing $G(x)$ by $F(s)$, we obtain

$$I_2 = \sqrt{2l}\int_0^d \frac{F(s)ds}{\sqrt{s}} \tag{2.4}$$

The integral of the right-hand member of (2.4) represents, by virtue of the second hypothesis, the one concerning independence of distribution of stresses and displacements in the end region, the universal characteristic coefficient of the material, introduced in [1], namely the cohesion modulus K.

Thus we have

$$I_2 = \sqrt{2l}\,K \tag{2.5}$$

Analogously we find that

$$I_3 = \int_a^{a+d} G(x)\sqrt{\frac{x-a}{b-x}}\,dx \approx \frac{1}{\sqrt{2l}}\int_0^d F(s)\sqrt{s}\,ds \tag{2.6}$$

Since

$$\int_0^d F(s)\sqrt{s}\,ds = \int_0^d \frac{F(s)}{\sqrt{s}}\,s\,ds < d\int_0^d \frac{F(s)ds}{\sqrt{s}} = Kd$$

we find

$$I_3 < \frac{Kd}{\sqrt{2l}}, \qquad \frac{I_3}{I_2} = O\left(\frac{d}{l}\right)$$

where $O(d/l)$ denotes a quantity of the order of magnitude of d/l; on the basis of the first hypothesis the integral I_3 can be disregarded compared with the integral I_2. Correspondingly, the first of relations (2.2) gives

$$\int_a^b p(x)\sqrt{\frac{x-a}{b-x}}\,dx = K\sqrt{2l} = K\sqrt{b-a} \tag{2.7}$$

Entirely analogously, the second of the relations (2.2) gives

$$\int_a^b p(x)\sqrt{\frac{b-x}{x-a}}\,dx = K\sqrt{2l} = K\sqrt{b-a} \tag{2.8}$$

Equations (2.7) and (2.8) determine the unknown coordinates of the ends of the crack. In particular, if the applied load is symmetrical with respect to $x = 0$, so that the crack is also symmetrical with respect to $x = 0$, i.e. $b = -a = l$, then conditions (2.7) and (2.8) become equivalent and assume the form

$$\int_{-l}^{l} p(x) \sqrt{\frac{l-x}{l+x}}\, dx = K \sqrt{2l}, \quad \text{or} \quad \int_{0}^{l} \frac{p(x)\, dx}{\sqrt{l^2 - x^2}} = \frac{K}{\sqrt{2l}} \tag{2.9}$$

3. Examples. Distribution of displacements over the surface of the crack. 1. Assume that the crack arises under the action of a constant pressure p_0 applied along a slit of length $2l_0$. The relation (2.9) then gives

$$p_0 \int_{-l_0}^{l_0} \sqrt{\frac{l-x}{l+x}}\, dx = 2p_0 l \arcsin \frac{l_0}{l} = K \sqrt{2l}, \tag{3.1}$$

Hence

$$\frac{p_0 \sqrt{l_0}}{K} = \frac{1}{\sqrt{2}} \sqrt{\frac{l_0}{l}} \frac{1}{\arcsin (l_0 / l)} \tag{3.2}$$

Figure 3 represents graphically the relation (3.2). The graph shows that equation (3.2) has no solution, if

$$p_0 < \frac{K}{\sqrt{l_0}} \frac{\sqrt{2}}{\pi} = p_0^*$$

This means that no open cracks can arise at so small values of p_0. Each $p_0 > p_0^*$ corresponds to a uniquely determined size of the crack; of course, the size of the crack increases with the increase of p_0.

2. Assume that the crack is produced by concentrated forces. This case arises if in the preceding example p_0 tends toward infinity, while l_0 is at the same time decreasing in such a way that the product $2p_0 l_0$ remains constant and equal to P, where P is the amount of the concentrated force. In this case we have

$$\frac{p_0 \sqrt{l_0}}{K} \approx \frac{1}{\sqrt{2}} \sqrt{\frac{l}{l_0}}, \quad \frac{2p_0 l_0}{K} = \sqrt{2l}, \quad \frac{P}{K} = \sqrt{2l}, \quad l = \frac{P^2}{2K^2} \tag{3.3}$$

It is not difficult to derive this result, with an accuracy to a constant factor, from considerations of dimensional analysis, making use of the so-called Π-theorem [6].

3. Assume now that the crack is produced by two concentrated forces of equal magnitude and opposite direction, whose points of application are situated at a distance $2L$ from each other along the common line of action

of these forces. We assume that for reasons of symmetry the crack will be perpendicular to the straight line joining the points of application of the forces and symmetrical with respect to it. Summation of the known fundamental solutions of the theory of elasticity [7] gives

$$p(x) = -\frac{P}{2\pi}\frac{L}{x^2+L^2}\left[-(3+\nu)+2(1+\nu)\frac{x^2}{x^2+L^2}\right] = \frac{P}{2\pi}\frac{L}{x^2+L^2}\left[1-\nu+2(1+\nu)\frac{L^2}{x^2+L^2}\right] \qquad (3.4)$$

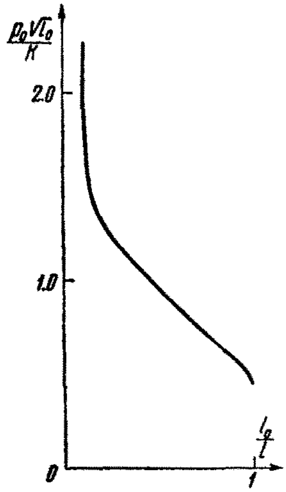

Fig. 3.

where x is the coordinate measured from the intersection point of the crack with the line of action of the forces along the crack. Substituting this expression into (2.9), we obtain

$$\int_{-l}^{l} p(x)\sqrt{\frac{l-x}{l+x}}\,dx = \frac{PL(1-\nu)}{2\pi}\int_{-l}^{l}\frac{dx}{x^2+L^2}\sqrt{\frac{l-x}{l+x}} + \frac{PL^3(1+\nu)}{\pi}\int_{-l}^{l}\frac{dx}{(x^2+L^2)^2}\sqrt{\frac{l-x}{l+x}} = K\sqrt{2l}$$

Evaluating the integrals, we find

$$\frac{P}{2}\left[2+\frac{L^2}{l^2}(3+\nu)\right]\left(\frac{L^2}{l^2}+1\right)^{-3/2} = K\sqrt{2l} \qquad (3.5)$$

It is convenient to transform this relation to

$$\frac{P}{K\sqrt{L}} = \left(\frac{L^2}{l^2}+1\right)^{3/2}\frac{\sqrt{2}}{[2+(3+\nu)L^2/l^2]\sqrt{L/l}} = u\left(\frac{L}{l}\right) \qquad (3.6)$$

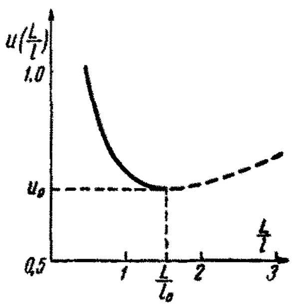

Fig. 4.

Figure 4 gives a graphical representation of the function (3.6) for the case $\nu = 0.5$. We see that the two roots of equation (3.6), i.e. two values of the crack length l, correspond to each value of P. It is, however, evident that, starting from the minimum point of the curve, no physical meaning can be assigned to values which correspond to the right-hand branch of that curve, because the size of the crack increases and the force decreases, so that the equilibrium states corresponding to this branch are unstable. Equation (3.6) has no solution for

$$\frac{P}{K\sqrt{L}} < u_0 = \frac{P_0}{K\sqrt{L}}$$

This means that for any L there is a corresponding critical value of the applied force, so that no equilibrium crack is possible at values of the applied force which are smaller than the critical value.

The results thus obtained are qualitatively identical with the results derived in [1] for the analogous cases of axisymmetrical equilibrium cracks.

4. It has been shown above that the function $\phi(\zeta)$, by means of which the displacements are expressible, has the form (we confine ourselves to the symmetrical case for reasons of simplicity)

$$\varphi(\zeta) = \frac{l}{2\pi i}\int_0^{\pi} g(l\cos\lambda)\sin\lambda \ln\left(\frac{e^{i\theta}-e^{i\lambda}}{e^{i\theta}-e^{-i\lambda}}\right)d\lambda = I_1 - I_2 - I_3 \qquad (3.7)$$

$$I_1 = \frac{l}{2\pi i}\int_0^{\pi} p(l\cos\lambda)\sin\lambda \ln\left(\frac{e^{i\theta}-e^{i\lambda}}{e^{i\theta}-e^{-i\lambda}}\right)d\lambda$$

$$I_2 = \frac{l}{2\pi i}\int_0^{\lambda_0} G(l\cos\lambda)\sin\lambda \ln\left(\frac{e^{i\theta}-e^{i\lambda}}{e^{i\theta}-e^{-i\lambda}}\right)d\lambda \qquad \left(\lambda_0 = \sqrt{\frac{2d}{l}}\right)$$

$$I_3 = \frac{l}{2\pi i}\int_{\pi-\lambda_0}^{\pi} G(l\cos\lambda)\sin\lambda \ln\left(\frac{e^{i\theta}-e^{i\lambda}}{e^{i\theta}-e^{-i\lambda}}\right)d\lambda \qquad (3.8)$$

Consider a point at the surface of the crack at such a distance from the ends of the latter which is large as compared with the width d of the end region; thus $\theta \gg \sqrt{d/l}$ and $\pi - \theta \gg \sqrt{d/l}$. For such θ and λ we have in the intervals $0 < \lambda < \sqrt{2d/l}$, $\pi - \sqrt{2d/l} < \lambda < \pi$

$$\ln\left(\frac{e^{i\theta}-e^{i\lambda}}{e^{i\theta}-e^{-i\lambda}}\right) = \ln\left(\frac{e^{i\theta}-1-i\lambda}{e^{i\theta}-1+i\lambda}\right) = \ln\left(\frac{1-i\lambda/(e^{i\theta}-1)}{1+i\lambda/(e^{i\theta}-1)}\right) = -\frac{2i\lambda}{e^{i\theta}-1} \qquad (3.9)$$

Substituting this for example, into the expression for I_2, we obtain

$$I_2 = -\frac{l}{\pi}\int_0^{\lambda_0}\frac{G(l\cos\lambda)\lambda^2 d\lambda}{e^{i\theta}-1} = \frac{1}{\pi\sqrt{l}(e^{i\theta}-1)}\int_{l-d}^{l} G(x)\sqrt{l-x}\,dx =$$

$$= \frac{1}{\pi\sqrt{l}(e^{i\theta}-1)}\int_0^d \frac{F(s)}{\sqrt{s}}\,s\,ds \qquad (3.10)$$

which shows that the absolute value of I_2 is of an order of magnitude not higher than $Kd/\sqrt{l}$. The same holds true of the integral I_3. The absolute value of I_1 is by virtue of (2.9) of the order of magnitude of $K\sqrt{l}$. So we see that both $|I_2|$ and $|I_3|$ are small compared with $|I_1|$. I_1 determines, however, such displacements of the surface of the crack, which correspond to its size determined by (2.9), but derived without taking into account the cohesive forces, while the quantities I_2 and I_3 determine the part of the displacements which is produced by the cohesive forces.

So we see that for points sufficiently far away from the ends of the crack the displacements produced by cohesive forces are, in analogy with the axisymmetrical case, small compared with those produced by the basic applied load. The displacements are essentially determined by the cohesive forces only in the vicinity of the ends of the crack; the latter circumstance explains the smooth connection of the opposite edges of the crack at the ends of the latter.

4. Problem of driving a wedge into a plate. 1. The problem of driving a wedge into a plate is formulated in the following manner* (Fig. 5). A rigid wedge of constant thickness $2h$ is driven into a plane plate of brittle material, whose modulus of elasticity, Poisson's ratio and modulus of cohesion are E, ν and K, respectively. The plate is assumed to be infinite, in other words, the influence of the boundary line is assumed to be negligible; in correspondence herewith the wedge is assumed to be semi-infinite. A slit of length L is formed in front of the wedge. It is required to determine the length L, as well as the stresses and deformations in the plate.

It follows from the preceding considerations that the size of the end region of any crack is of the order of magnitude of K^2/E^2; assume that h is large as compared with the size of the end region; then the length L of the formed crack is large as compared with h, and the problem can be linearized by transferring the boundary conditions from the surface of the crack to the axis Oy (the origin of coordinates is placed, for reasons of convenience, at the point of joining of the edges of the crack). A similarly conceived statement of the problem is encountered in the study of deflection of a smoothly lowered roof into a coal mine. The solution of the latter problem was given in [3], where the cohesive forces of the material are not taken into consideration. The cohesive forces are, however, of primary importance in the problem considered in the present paper, since other compressing factors of the type of rock pressure do not occur in our present problem. We shall see in the following that the necessity of taking the cohesive forces into account introduces into our present problem several particular features.

If the frictional forces on the surface of the wedge are disregarded, then the boundary conditions along the cut $0 \leqslant y < \infty$ have for the problem

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* The problem is visualized by the case of an axe driven into a log without splitting it entirely.

under consideration the form

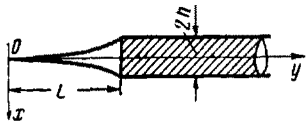

Fig. 5.

$$0 \leqslant y \leqslant d, \qquad X_y = 0, \qquad X_x = G(x) \tag{4.1}$$

$$d < y \leqslant L, \qquad X_y = 0, \qquad X_x = 0 \tag{4.2}$$

$$L < y < \infty, \quad X_y = 0,$$

$$u\,(x+0,\ y) = h, \qquad u\,(x-0,\ y) = -h \tag{4.3}$$

where d is the width of the end region, X_x, Y_y, X_y are the components of the stress tensor, $G(x)$ represents the distribution of the cohesive forces in the end region and u, v are the displacement components along the x- and y-axes, respectively.

The problem with boundary conditions (4.1) to (4.3) is obviously a mixed one. Introducing the compressive stress resultants, acting on the faces of the wedge, and putting $X_x = -f(y)$, we replace the condition (4.3) by a condition of the first kind

$$L < y < \infty, \qquad X_y = 0, \qquad X_x = -f(y) \tag{4.4}$$

and the problem with boundary conditions (4.1), (4.2), (4.4) becomes a problem of the first kind. The function $f(y)$ is, however, unknown beforehand; to obtain this function we have to construct a singular integral equation and to solve it.

2. We use the method of Muskhelishvili [4] for the solution of the elastic problem with boundary conditions (4.1), (4.2), (4.4). Let us start with the relations

$$X_x + Y_y = 4\,\mathrm{Re}\,\{\Phi(\zeta)\} \tag{4.5}$$

$$Y_y - X_x + 2iX_y = 2\left\{\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\Phi'(\zeta) + \Psi(\zeta)\right\} \tag{4.6}$$

$$2\mu\,(u + iv) = \varkappa\varphi(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi'(\zeta)} - \overline{\psi(\zeta)}, \qquad \varkappa = 3 - 4\nu \tag{4.7}$$

in which

$$\Phi(\zeta) = -\frac{1}{2\pi i\zeta}\int_{-\infty}^{\infty}\frac{F\,d\sigma}{\sigma-\zeta}, \qquad \Psi(\zeta) = \frac{\zeta}{2}\Phi'(\zeta), \qquad F(\sigma) = X_x(\sigma)\,\sigma$$

$$\varphi(\zeta) = \int_0^{\zeta}\omega'(\zeta)\,\Phi(\zeta)\,d\zeta, \qquad \psi(\zeta) = \int_0^{\zeta}\omega'(\zeta)\,\Psi(\zeta)\,d\zeta \tag{4.8}$$

where μ is the shear modulus of the medium, while ν is Poisson's ratio; the function

$$z = i\zeta^2 = \omega(\zeta) \tag{4.9}$$

effects the mapping of the physical plane $z = x + iy$ with a cut along the positive imaginary semi-axis on the lower semi-plane of the parametric variable ζ.

Using (4.6) and (4.8) we find

$$Y_y - X_x + 2iX_y = \frac{\Phi'(\zeta)}{\zeta}(\zeta^2 - \bar{\zeta}^2) \tag{4.10}$$

such that for real and purely imaginary ζ, corresponding to the crack and its continuation, we again obtain relations (1.7) and (1.8). Furthermore, using relations (4.7) and (4.8), we have for real and purely imaginary ζ, i.e. on the crack itself and its continuation,

$$2\mu(u + iv) = \varkappa\varphi(\zeta) + \overline{\varphi(\zeta)} \tag{4.11}$$

Thus, for the determination of the stresses and displacements at the crack itself and its continuation, it is sufficient to know the function $\phi(\zeta)$, and this means to know the function $\Phi(\zeta)$ as well, which is connected with the function $\phi(\zeta)$ in a very simple manner.

According to boundary conditions (4.1), (4.2), (4.4) and relation (4.8) we have

$$\varphi'(\zeta) = 2i\zeta\Phi(\zeta) =$$
$$= \frac{1}{\pi}\int_{-\infty}^{-\sqrt{L}} \frac{f(\sigma^2)\,\sigma\,d\sigma}{\sigma - \zeta} + \frac{1}{\pi}\int_{\sqrt{L}}^{\infty} \frac{f(\sigma^2)\,\sigma\,d\sigma}{\sigma - \zeta} - \frac{1}{\pi}\int_{-\sqrt{d}}^{\sqrt{d}} \frac{G(\sigma^2)\,\sigma\,d\sigma}{\sigma - \zeta}. \tag{4.12}$$

According to the third hypothesis, the hypothesis of Khristianovich, the stress must be finite at the end of the crack, i.e. for $\zeta = 0$. It follows that $\Phi(0)$ must be finite, so that $\phi'(0)$ must equal zero. This, together with (4.12), gives

$$\frac{1}{\pi}\int_{-\infty}^{-\sqrt{L}} f(\sigma^2)\,d\sigma + \frac{1}{\pi}\int_{\sqrt{L}}^{\infty} f(\sigma^2)\,d\sigma - \frac{1}{\pi}\int_{-\sqrt{d}}^{\sqrt{d}} G(\sigma^2)\,d\sigma =$$
$$= \frac{2}{\pi}\int_{\sqrt{L}}^{\infty} f(\sigma^2)\,d\sigma - \frac{2}{\pi}\int_{0}^{\sqrt{d}} G(\sigma^2)\,d\sigma = 0$$

It is, however, evident that

$$\int_{0}^{\sqrt{d}} G(\sigma^2)\,d\sigma = \int_{0}^{d} \frac{G(s)\,ds}{2\sqrt{s}} = \frac{1}{2}K$$

where K is the cohesion modulus of the material. So we have the condition for obtaining finite stresses at the end of the crack in the form

$$\int_{\sqrt{L}}^{\infty} f(\sigma^2)\,d\sigma = \frac{1}{2}K \tag{4.13}$$

or finally in the form

$$\int_{L}^{\infty} \frac{f(y)\,dy}{\sqrt{y}} = K \tag{4.14}$$

It can be shown that the same condition (4.11) is the condition for smooth connection of the opposite edges of the crack at the end of the latter.

3. Formula (4.11) gives

$$u = \frac{\varkappa+1}{2\mu}\operatorname{Re}\{\varphi(\zeta)\} = \frac{4(1-\nu^2)}{E}\operatorname{Re}\{\varphi(\zeta)\} \tag{4.15}$$

Integration of (4.12) yields

$$\varphi(\zeta) = \frac{1}{\pi}\int_{\sqrt{L}}^{\infty} f(\sigma^2)\,\sigma\ln\left|\frac{\sigma+\zeta}{\sigma-\zeta}\right|d\sigma - \frac{1}{\pi}\int_{0}^{\sqrt{d}} G(\sigma^2)\,\sigma\ln\left|\frac{\sigma+\zeta}{\sigma-\zeta}\right|d\sigma. \tag{4.16}$$

We shall now show that in the case under consideration the distribution of the displacements not in the immediate vicinity of the end of the crack, i.e. at such points of the surface of the latter whose distance from the end of the crack is large compared with the size of the end region, is again independent of the cohesive forces. If a point of the crack surface is not in immediate vicinity of the end of the crack, then $y \gg d$, so that the value of ζ corresponding to this point is large in magnitude compared with $\sqrt{d}$. Under these conditions

$$\int_{0}^{\sqrt{d}} G(\sigma^2)\,\sigma\ln\left|\frac{\sigma+\zeta}{\sigma-\zeta}\right|d\sigma \approx \frac{2}{\zeta}\int_{0}^{\sqrt{d}} G(\sigma^2)\,\sigma^2 d\sigma < \frac{Kd}{\zeta} \ll K\sqrt{d}$$

The value of the function $\phi(\zeta)$ not in the immediate vicinity of the end of the crack is of the order of magnitude of Eh. It is, however, evident that $Eh \gg K\sqrt{d}$, since according to the preceding discussion E/K is of the order of magnitude of $\sqrt{d}$ and $h \gg d$. Consequently, the second integral of the right-hand member of (4.16) (this is the integral which determines the dependence between the displacements and the cohesive forces at points not in the immediate vicinity of the end of the crack, i.e. at points whose distances are of the order of magnitude of several d's from the end of the crack) is small and can be neglected. This applies in particular to all points of contact between the rigid wedge and the wedged plate. For such points we have

$$u = \frac{4(1-\nu^2)}{\pi E} \int\limits_{\sqrt{L}}^{\infty} f(\sigma^2)\, \sigma \ln\left|\frac{\sigma+\zeta}{\sigma-\zeta}\right| d\sigma = \pm h \tag{4.17}$$

where the plus or minus sign is to be used depending on whether ζ is positive or negative, i.e. whether ζ corresponds to the lower or upper edge of the cut.

In complete analogy with [3], the following result can be derived: in order to obtain a displacement of absolute value equal to h for $y \to \infty$, the asymptotic equation

$$f(y) = \frac{Eh}{2\pi(1-\nu^2)\, y} + O(y^{-2}) \tag{4.18}$$

must be fulfilled for $y \to \infty$.

Furthermore, differentiating (4.17) with respect to ζ we obtain a singular integral equation for $f(y)$ of the form

$$\int\limits_{\sqrt{L}}^{\infty} f(\sigma^2)\, \sigma \left\{\frac{1}{\sigma+\zeta} + \frac{1}{\sigma-\zeta}\right\} d\sigma = 0. \tag{4.19}$$

which can be written also in the form

$$\int\limits_{|\sigma| > \sqrt{L}} \frac{f(\sigma^2)\, \sigma\, d\sigma}{\sigma-\zeta} = 0 \tag{4.20}$$

The integral equation (4.20), taken together with condition (4.18), is equivalent to the integral equation (4.17). Substituting

$$\sigma = \frac{1}{\tau}, \qquad \zeta = \frac{1}{\xi}, \qquad p(\tau) = \frac{1}{\tau^2} f\left(\frac{1}{\tau^2}\right)$$

into (4.20), we obtain

$$\int\limits_{-\Lambda}^{+\Lambda} \frac{p(\tau)\, d\tau}{\tau-\xi} = 0 \qquad \left(\Lambda = \frac{1}{\sqrt{L}}\right) \tag{4.21}$$

Mikhlin has shown in connection with another problem [8] that the solution of (4.21) is of the form

$$p(\xi) = A\left[\frac{1}{L} - \xi^2\right]^{-1/2} \tag{4.22}$$

where A is an arbitrary constant; so we find

$$f(y) = \frac{A\sqrt{L}}{\sqrt{y}\sqrt{y-L}} \tag{4.23}$$

If $y \to \infty$, then relation (4.23) leads to

$$f(y) = \frac{A\sqrt{L}}{y} + O(y^{-2}) \tag{4.24}$$

Compare this asymptotic representation with that given in (4.18); the latter is the condition for the fulfilment of the requirement that the displacement have an absolute value equal to h at infinity. The comparison leads to

$$A\sqrt{L} = \frac{Eh}{2\pi(1-\nu^2)} \tag{4.25}$$

The stress distribution (4.25) must also fulfil the condition that the stress be finite at the end of the crack, see (4.14), whence

$$\int_L^\infty \frac{f(y)\,dy}{\sqrt{y}} = A\sqrt{L}\int_L^\infty \frac{dy}{y\sqrt{y-L}} = A\int_1^\infty \frac{dz}{z\sqrt{z-1}} = \pi A = K \tag{4.26}$$

so that

$$A = \frac{K}{\pi} \tag{4.27}$$

From (4.25) and (4.27) we obtain the expression for the length L of the slit

$$L = \frac{E^2h^2}{4(1-\nu^2)^2K^2} \tag{4.28}$$

Thus, knowing the function $f(y)$, we can determine the functions $\phi(\zeta)$ and $\Phi(\zeta)$ from (4.12) and (4.16), after which we can obtain all components of stress and displacement from relations (4.5) to (4.7).

We believe that formula (4.28) can be used as a basis for one of the convenient ways of determining the cohesion modulus K. It is sufficient to this end to drive into a plate of the material under examination a wedge of constant width $2h$ and made of a material essentially harder than that of the plate (e.g. a steel wedge can be used for determining the cohesion modulus of plexiglass*). The wedge must be driven so far as to make the distance L between the end of the wedge and the end of the crack in front of the wedge constant, showing that the influence of the boundaries of the plate is inessential. This length L must be measured. Knowing Young's modulus and Poisson's ratio of the material under consideration, we can determine its cohesion modulus K by means of formula (4.28).

A preliminary experiment of this kind was carried out for the case of plexiglass by Maraev in the laboratory of Geiman**. A wedge made of a plane steel spring of thickness $2h$ = 0.034 cm was driven into a plate of plexiglass. A nearly rectilinear crack arose of length L = 2 cm. Assuming that Young's modulus for plexiglass equals to E = 25,000 kg/cm^2, while

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* Trade mark for cast acrylic resin thermoplastic sheets and moulding powder.

** The author wishes to express his gratitude to Geiman and Maraev.

Poisson's ratio is $\nu = 0.25$, we obtain for the cohesion modulus the value

$$K = \frac{Eh}{2(1-\nu^2)\sqrt{L}} \approx 160 \quad \text{kg/cm}^2$$

A cohesion modulus of approximately 100 kg/cm$^{3/2}$ was obtained for another sample of plexiglass. The cohesion modulus K is of the order of magnitude of $E\sqrt{d}$, where d characterizes the size of the end region; it follows that d is of the order of 1 μ (10^{-4}cm), i.e. the size of the end region is small as compared with the entire length of the crack, but large in comparison with interatomic distances (10^{-8}cm).

We again emphasize that the experiment just mentioned was one of preliminary nature. A careful experimental verification of the concepts developed in [1], as well as in the present paper, would be very desirable.

5. Remark on possibilities of taking into account the cohesive forces and the influence of the boundaries of the solid on the development of cracks. 1. It was shown above that the cohesive forces are substantially influencing only the size of the crack and the distribution of stresses and displacements in the immediate vicinity of its ends. Thus, having determined the dimensions of the crack, it is possible to treat the problem as a problem of the theory of elasticity for a solid with cracks, disregarding the cohesive forces. With this procedure we will arrive at stresses and displacements, particularly at displacements of points of the surface of the crack, which nearly equal, everywhere except the vicinity of the ends of cracks, the corresponding values obtained in calculations which take cohesive forces into account.

Consider for example an isolated rectilinear crack in an infinite solid. If the cohesive forces are disregarded, then according to the foregoing the stress components X_x and Y_y in the vicinity of the end $x = b$ approach infinity according to the law

$$X_x = X_y = \frac{1}{\pi\sqrt{2(x-b)\,l}} \int_a^b p(x) \sqrt{\frac{x-a}{b-x}}\, dx + \ldots, \qquad l = \frac{1}{2}(b-a) \quad (5.1)$$

where the points indicate quantities of higher order of smallness. By virtue of (2.8), expression (5.1) assumes the form

$$X_x = Y_y = \frac{1}{\pi\sqrt{s}} K, \qquad s = x - b \quad (5.2)$$

An analogous relation holds true in the vicinity of $x = a$. This result is of very general significance in the following sense. The theory of elasticity permits to determine the state of stress for any position of the crack ends and for any given loading, if the cohesive forces are disregarded; the stresses near the ends tend toward infinity according to the law $A/\sqrt{s}$, where s is the distance from the end of crack and A is a constant, in general varying for various ends, and depending on the position of the latter. We can state as a general rule: *the ends of a crack are determined from the condition that the stresses in their vicinity, computed without taking cohesive forces into consideration, tend to infinity according to the law*

$$K / \pi \sqrt{s} \tag{5.3}$$

This rule permits in general to exclude from the consideration the cohesive forces as such.

2. We illustrate the procedure of taking the cohesive forces into account indicated above by means of a problem, which is also of interest by itself, from the point of view of determining the influence of the boundaries of a solid on the development of cracks.

Assume that a crack is produced in an infinite strip under the action of forces P, of equal magnitude and opposite direction, applied to the surface of the crack. The width of the strip is $2L$, the forces are acting along the center line of the strip (Fig. 6).

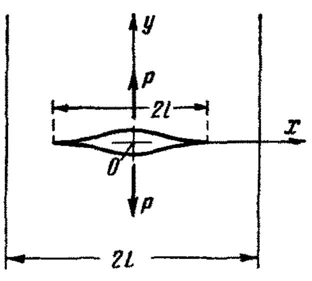

Fig. 6.

This problem is solved by means of the method of successive approximations developed by Mikhlin [9] and Sherman [10]. It is convenient to take as a first approximation the stress field in the strip $(-L \leqslant x \leqslant L,$ $-\infty < y < \infty)$ of the infinite solid represented by the exterior of the periodic system of cracks situated along the x-axis, with centers at points $x = \pm 2nL$ (n = an integer). The loadings which produce the cracks are the same as for the strip under consideration. Only the shear stresses vanish along the lines $x = \pm L$; the normal stresses do not vanish there. To compute the second approximation, we have to consider the problem of a continuous strip under compression by boundary stresses; these stresses are of magnitude equal to that of, and opposite in direction to, the normal stresses obtained along the boundaries in the first approximation. In this procedure some normal stresses will arise along the x-axis; for the elimination of these stresses and the derivation of the third approximation, it is again necessary to solve some periodical problem, and so forth. It has been shown, however, by Irwin [11], that the normal stresses along the boundaries of the strip $x = \pm L$ are of comparatively small influence on the propagation of cracks normal to the boundaries, therefore, we confine ourselves to the first approximation.

Using the results obtained by Irwin [11], we can show that for the stresses along the x-axis we obtain, disregarding cohesive forces, in the first approximation

$$X_x = Y_y = \frac{1}{2L \sin(\pi x / 2L)} P \sin\frac{\pi l}{2L}\left[\sin^2\left(\frac{\pi x}{2L}\right) - \sin^2\left(\frac{\pi l}{2L}\right)\right]^{-1/2}, \quad X_y = 0 \qquad (5.4)$$

In the vicinity of the points $x = \pm (l + s)$, where s is a small quantity, we obviously have

$$X_x = Y_y = \frac{1}{\sqrt{2\pi L \sin(\pi l / L)}} \frac{P}{\sqrt{s}} \qquad (5.5)$$

Comparing (5.3) and (5.5) with each other we find

$$\frac{P}{\sqrt{2\pi L \sin(\pi l / L)}} = \frac{K}{\pi} \quad \text{or} \quad \frac{L}{\pi} \sin\frac{\pi l}{L} = \frac{P^2}{2K^2} \qquad (5.6)$$

In the case of $l \ll L$ we again obtain formula (3.4) for the length of the crack in an infinite plate. The function (5.6) is represented graphically in Fig. 7 (curve *II*; curve *I* corresponds to the transformed formula (3.3)). We see that in contrast to curve *I*, curve *II* has an unstable part

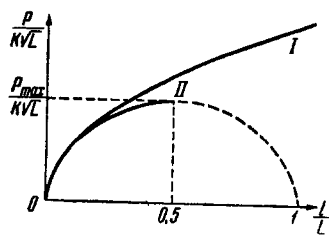

Fig. 7.

(shown by dotted line), along which the force necessary to keep the crack in equilibrium condition is decreasing, while the length of the crack increases. In other words, after the load has reached its maximum value

$$P_{\max} = \sqrt{\frac{2}{\pi}} K \sqrt{L} \qquad (5.7)$$

even the smallest load increase leads to a sudden widening of the crack to an amount equal to the width of the strip and to a destruction of the latter.

We note that the situation becomes more complicated if the forces are applied not at points of the surface of the crack, but at some distance from each other along the center line of the strip. Namely, if the distance of the points of application of the forces from each other is smaller than a certain critical value, then a crack is in general not formed until the force has reached a certain magnitude. As soon as the force reaches this magnitude, a crack of a certain finite size develops; on further increasing the force, the crack gradually increases until a certain possible maximum load is reached. Even the slightest increase of the latter amount causes sudden destruction of the strip.

If the distance of the points of application of the forces is larger than the critical distance, then no crack will develop in general, until the force has reached a certain definite value. As soon as the force reaches this limiting value, the strip is suddenly torn; no stable equilibrium cracks are possible in this case.

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